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From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory

By Jean-Pierre Marquis. Logic, Epistemology, and the Unity of Science. (Springer). 2010. ISBN 978-9048181179. 320 pp. $199.

Felix Klein's 1872 Erlangen Program has always been more famous than familiar. In fact, Klein's greatness lay less in his own mathematics than in his ability to recognize and promote the best new mathematics of his time. It is natural that his program be known for its overall vision more than for the particulars.

The program grew from many sources including Plücker's projective geometry, Lie's symmetry groups, Clebsch on algebraic invariants, and the rise of topology under the name *analysis situs*. Projective geometry and invariant theory faded from prominence over the next decades while topology, Lie groups, and wider ideas of symmetry and transformation exploded, notably encouraged by general relativity and quantum mechanics.

The high opinion of Klein's program today reflects that later trend. And this is entirely fair. Klein never suggested limiting geometry to the methods he knew. He called attention to the rise and the unity of a broad family of great new methods; and he had a synoptic vision of those methods far ahead of anyone else.

Eilenberg and Mac Lane gave a nice historical nod in 1945 in their first paper on general category theory, saying "This may be regarded as a continuation of Klein's *Erlanger Programm*, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings" (quoted on p. 9). People have wondered ever since precisely what they meant, and Marquis makes that the start for a wide-ranging reflection on category theory.

Marquis has achieved two things. One is a very interesting history and philosophy of category theory accessible to readers with little knowledge of category theory yet valuable to anyone interested in the subject. The other is to open several lines of inquiry into just how category theory relates to Klein's program. The upshot gives Klein no very specific role in fact. Marquis argues that Eilenberg and Mac Lane "did not extend Klein's program as such, although they clearly made an effort to extend a part of it" (p. 4), and even this claim must be qualified since "the connection with Klein's program probably came as an afterthought" (p. 66). Most precisely he puts his "central claim" this way: "with hindsight one can argue that Klein's program is one very special case of the power, richness, and persuasiveness of categorical methods" (p. 3).

He also notes historical links from Klein to Eilenberg and Mac Lane (p. 3): Mac Lane studied in Göttingen in 1932–33 when the Mathematics department was still very much Klein's creation. He heard lectures by Hilbert and Noether who both became prominent under Klein's aegis and came to Göttingen on his initiative. Eilenberg's work in algebraic topology already in the 1930s meant he studied the work of several students of Noether.

Marquis especially emphasizes the geometric motivations for much of category theory and the in-some-ways geometric nature of categories themselves. It is a commonplace of history of mathematics to say that geometry went out of fashion in the 20th century. The usual evidence cited is the absence of geometry from Bourbaki's *Elements of Mathematics*, and a claim that Bourbaki helped shift mathematics education in France away from geometry. But at the same time Bourbaki members did a huge amount of geometry themselves. As a notable example André Weil generalized the Gauss–Bonnet theorem (relating the total curvature of a surface to its topology) so elegantly and so greatly as to produce the subject of *connections on fiber bundles* now central to quantum field theory.

Weil's work extended Élie Cartan's geometry of group actions on spaces which Marquis discusses at length as a 20th century successor to Klein's program. These ideas spread all over differential geometry and mathematical physics. Several members of Bourbaki, including Weil and then Grothendieck, brought geometric methods into number theory culminating (so far) in *arithmetic algebraic geometry*.

The geometrization of physics and number theory have been prestigious archetypes of 20th century mathematics since the opening of the century with Poincaré and Hilbert; through mid-century with Lefschetz, Weyl, and von Neumann (besides those named above); and beyond the century's end with Witten and the Langland's program. These geometrizations were and still are major sources of category theory and today their tools are pervasively categorical. Marquis's last chapter describes how even logic has been geometrized using related tools. This book should go a long way to dispel the idea that geometry was out of fashion in the 20th century.

Marquis gives no concise statement of the goal of the Erlangen program and this is quite reasonable. Klein himself did not pose a goal. He described an array of then-recent results in geometry which he organized around this "general problem":

> Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group. [1893, p. 218]

For a familiar example of transformation groups, two figures are *congruent* in the Euclidean plane if one can be carried over to the other by rigidly translating, rotating, or reflecting the plane. So the transformation group of Euclidean plane congruence-geometry is the group of translations, rotations, and reflections. On the other hand, two figures are *similar* if one can be carried over to the other by rigidly translating, rotating, reflecting, or scaling up or down. So the transformation group of Euclidean plane similarity-geometry is the group of all translations, rotations, reflections, and scalings. The geometries can be studied in terms of their transformation groups.

Marquis illustrates Klein's point that apparently different geometries can have the same transformation group. He gives a "philosophical fable" in which three mathematicians study apparently different geometries: one geometry of circles in the plane, one of great circles on a sphere, and one of the complex projective plane. These are "visually" different, so to speak, but Marquis shows how they all have "the same" transformation group and in that sense they are merely different ways of looking at the "same" geometry. All problems in any one of these geometries can be solved by translating them into any of the others and solving them there.

Focus on transformation groups had been brilliantly successful in non-Euclidean geometry. Klein and others showed that Euclidean, hyperbolic, and spherical plane geometry are *sub-geometries* of projective plane geometry in the sense that the transformation group of each is a subgroup of the symmetries of the projective plane. Euclidean affine geometry corresponds to the subgroup which leaves a specified *line at infinity* invariant. Intuitively, we take the points which are *not* on that line to be the "finite points", so that an affine transformation is linear and maps all finite points to finite points. Euclidean similarity geometry corresponds to the subgroup which leaves the line at infinity invariant together with a specified *absolute involution* of that line. Intuitively it is affine and preserves angles. Euclidean congruence geometry, and hyperbolic, and spherical geometries correspond to other subgroups In this way the points of Euclidean, hyperbolic, and spherical plane geometry can all be taken to be (among) the points of the projective plane.

This was a very pretty link between projective geometry and several kinds of Euclidean geometry. And, far beyond a mere logical consistency proof, this method gave hyperbolic and spherical geometries great legitimacy by linking them to projective geometry. Marquis cites many sources discussing this.

Klein takes for granted that a "manifoldness" is a projective space—so the geometries are all subsets of projective spaces (of any dimension). This is like the assumption common well into the 20th century that differential manifolds are subsets of real coordinate spaces $\mathbb{R}^n$. He stresses that the transformation groups, though, need not be contained in the linear groups naturally associated to projective spaces. This is vital to Marquis's topic.

Klein discusses how analysis situs goes beyond linear transformations to all continuous transformations. He also considers more complicated *contact transformations*. Only the simplest kind of those is much known today, namely the dualities in projective geometry which transforms points to higher dimensional subspaces and vice versa. These more general geometries have transformation groups so different from the projective geometries that Klein has little to say about them.

He specifically urges geometers to go beyond the projective viewpoint (and build a greater unity with physics) by working "the rich mine of mathematical truths brought to light by the theory of the curvature of surfaces" [1893, p. 244] and he mentions Lie's progress on related problems. But Lie used a *local* notion of transformation, which does not transform an entire space into itself, but only a part of it. In the 20th century Élie Cartan would organize and extend this a very great deal and even his methods would not work for spaces with irregularly non-constant curvature. Those spaces generally have no non-trivial symmetries even locally because the curvature is not even locally symmetric.

Marquis relates all of this to category theory in several ways. Eilenberg and Mac Lane's claim is certainly true on its face. Where Klein would study a single geometric space by its "group of transformations" to itself, they would study a family of objects (say, the class of all Abelian groups, or of all topological spaces) by the "algebra of mappings" between them (quoted on p. 9). But are these two projects really so similar? Klein really succeeded only with the very special case of linear transformation groups acting on (subsets of) projective spaces, which is remote from Eilenberg and Mac Lane's stated concerns.

Eilenberg and Mac Lane looked not at one space at a time and its transformations to itself; but at the category of all spaces (of any one kind) at once and all the transformations among them. And, crucially, Klein took transformations to be invertible or as we now say isomorphic—essentially what was called "change of coordinates." Klein explicitly took it that two spaces with such a transformation between them are "the same" in the relevant sense. Eilenberg and Mac Lane relied on transformations or mappings between explicitly quite different spaces—for example a map from the line to the circle which wraps the line infinitely many times around the circle. The line and the circle are in no relevant sense the same space.

Much further, Eilenberg and Mac Lane focused on "transformations" from one entire category to another, called *functors*. Klein did in fact use analogous group homomorphisms from the transformation group of one space to that of another. If we use categorical hindsight then we will say he could hardly avoid doing that. It was implicit in his reasoning, though only the case of one group being included in another got explicit attention.

Yet the broader idea was implicit, and Klein explicitly called for further advances. He succeeded as he intended at emphasizing programs productive in his own time which would continue on far beyond anything imaginable then. His overall thought cohered with actual trends in early 20th century pure mathematics and in physics so that Cartan and others had Klein's program in mind as they went far beyond the program's own framework and this is still going on today. Marquis shows how this led to standard geometric techniques today whereby every closed normal subgroup of any Lie group defines a space—including Klein's spaces as very special cases. Marquis discusses the concepts notably in his twin sections "Why a Transformation Group is Not Quite Enough" and "But Then Again, why a Group is Enough," and in a section on philosophical concepts of supervenience and reduction. He well says "This progression from 'concrete' representations to abstract structures will accompany us throughout, for it is a key feature of the whole of 20th century mathematics" (p. 30). It remains key a good decade into the 21st.

Marquis's book is well read together with Krömer [2007]. They have somewhat complementary time ranges as Marquis has more on the 19th and early 20th century background, and on recent ideas of higher dimensional categories, while Krömer has more on mid-20th century topology and especially Grothendieck. Marquis focusses more on philosophic aspects of geometrization and notions of identity versus isomorphism and equivalence. Krömer uses more archival research into unpublished sources and his philosophy is more aimed at pragmatism and issues of philosophy of science. Both make good use of interviews with living sources. Both cover many viewpoints on current debate over categorical foundations for mathematics.

Much of this book builds towards the account of *higher dimensional* categories. A category can be seen as a network of (one dimensional) arrows linking (zero dimensional) points. This technically useful description is used intuitively throughout the literature of pure category theory and applications. A 2-category further has 2-cells between arrows, visualized as surfaces with the arrows as edges. Higher dimensional categories may have $(n+1)$-cells going between n-cells for any $n \in \mathbb{N}$. Non-trivial examples occur throughout category theory and are central to much recent homotopy theory in topology. They make an intriguing link of category theory with geometry, but all require more explanation than we will give here.

There is a crucial difference in that category theory leads to *strict* n-categories while the rising connections with topology use *weak* n-categories. There is a long-established standard definition of the strict case. There is so far no accepted definition of weak n-category. There are many candidates. With time the best of them may be selected and unified; or truly different candidates may prove best for different purposes. Marquis describes the mathematical motives. He argues that the weak n-categorical viewpoint may lead to replacing the notions of equality or identity by weaker notions of isomorphism of objects (and thus eliminating isomorphism of categories in favor of equivalence of categories). And he speculates on ways this could, or should, affect thinking about categorical foundations for mathematics. But these arguments remain somewhat up in the air for now, because both "a proper characterization of the so-called weak *n*-categories" and "a proper axiomatization" of a foundation for mathematics in these terms "are still open problems at the moment of writing" (p. 7).

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